ENGINEERING SURVEYING 1

HORIZONTAL CURVES

CIRCULAR CURVES, COMPOUND CIRCULAR CURVES, REVERSE CIRCULAR CURVES

TRANSITION CURVES AND COMPOUND CURVES

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1. TYPES OF HORIZONTAL CURVES

The types of horizontal curves usually encountered in engineering surveying application may be broadly categorised as

(i) *Circular curves:* curves of constant radius joining two intersecting straights.

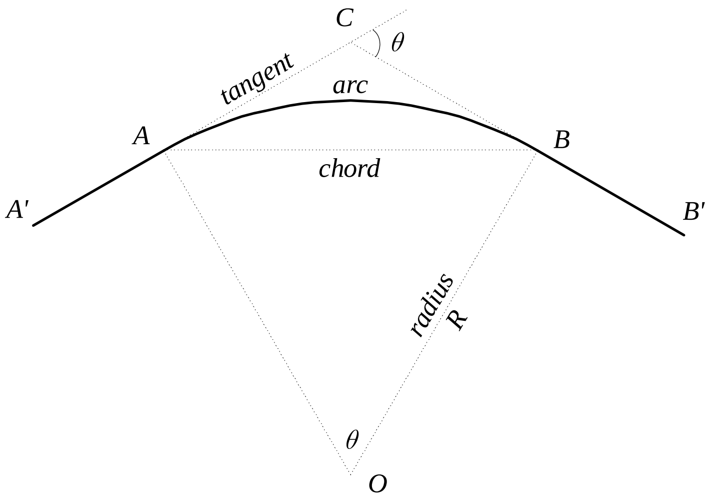


Figure 1.1

In Figure 1.1, a circular curve of constant radius *R*, centred at *O*, joins two straights *A'A* and *BB'* which intersect at *C*. A and *B* are tangent points to the circular arc. *OA* and *OB* are radials, which meet the straights at right angles, and the angle at *O* is equal to the intersection angle at *C*.

(ii) *Compound circular curves:* two or more consecutive circular curves of different radii.

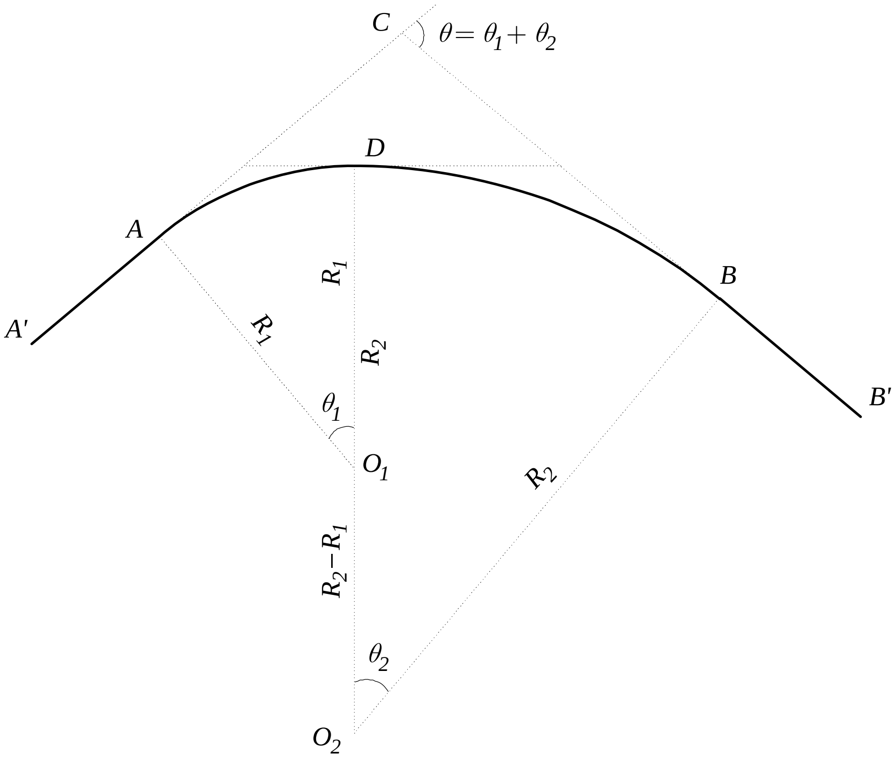


Figure 1.2

In Figure 1.2, a compound circular curve *ADB* joins two straights *A'A* and *BB'* which intersect at *C*. A and *B* are tangent points to circular arcs of radii R_1 and R_2 respectively. *D* is a common tangent point.

(iii) *Reverse circular curves:* two or more consecutive circular curves, of the same or different radii whose centres lie on different sides of a common tangent point.

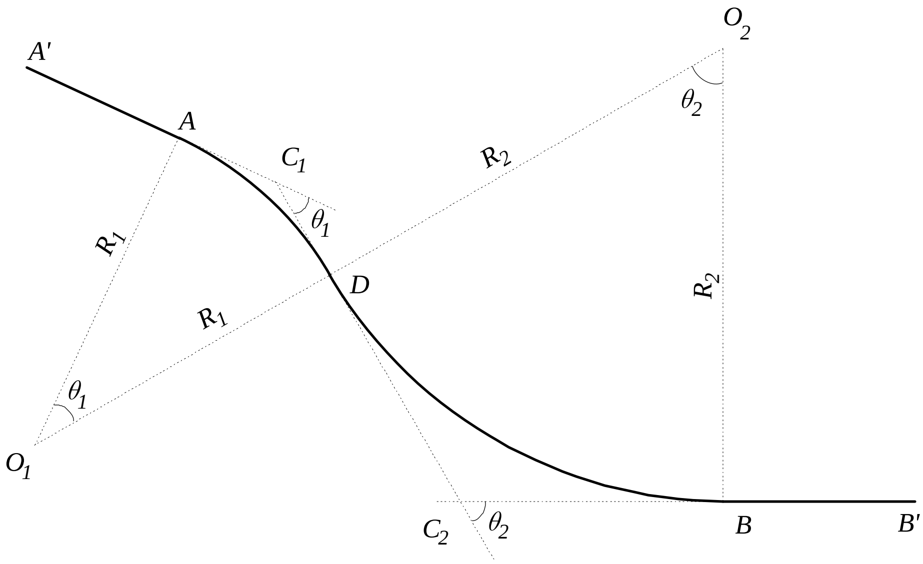


Figure 1.3

In Figure 1.3, a reverse circular curve *ADB* joins two straights *A'A* and *BB'*. A and *B* are tangent points to circular arcs of radii R_1 and R_2 respectively. *D* is a common tangent point. C_1 and C_2 are intersection points and the line C_1C_2 is perpendicular to the line between the centres O_1 and O_2 .

(iv) *Transition curves:* curves with gradually changing radius, often referred to as spirals.

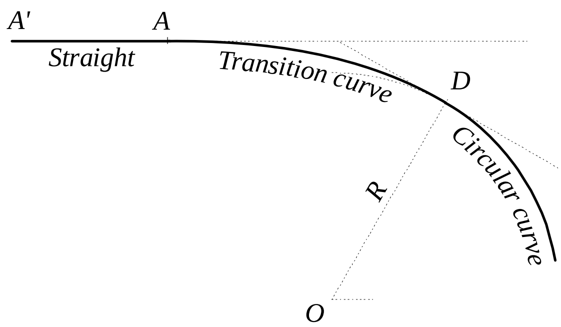


Figure 1.4

In Figure 1.4, a transition curve *AD* joins the straight *A'A* and the circular curve of radius *R* whose centre is *O*. The transition curve has an infinite radius at *A*, decreasing gradually to a radius of *R* at *D*.

(v) *Combined curves:* consisting of consecutive transition and circular curves. Combined curves are used in road and railway surveying.

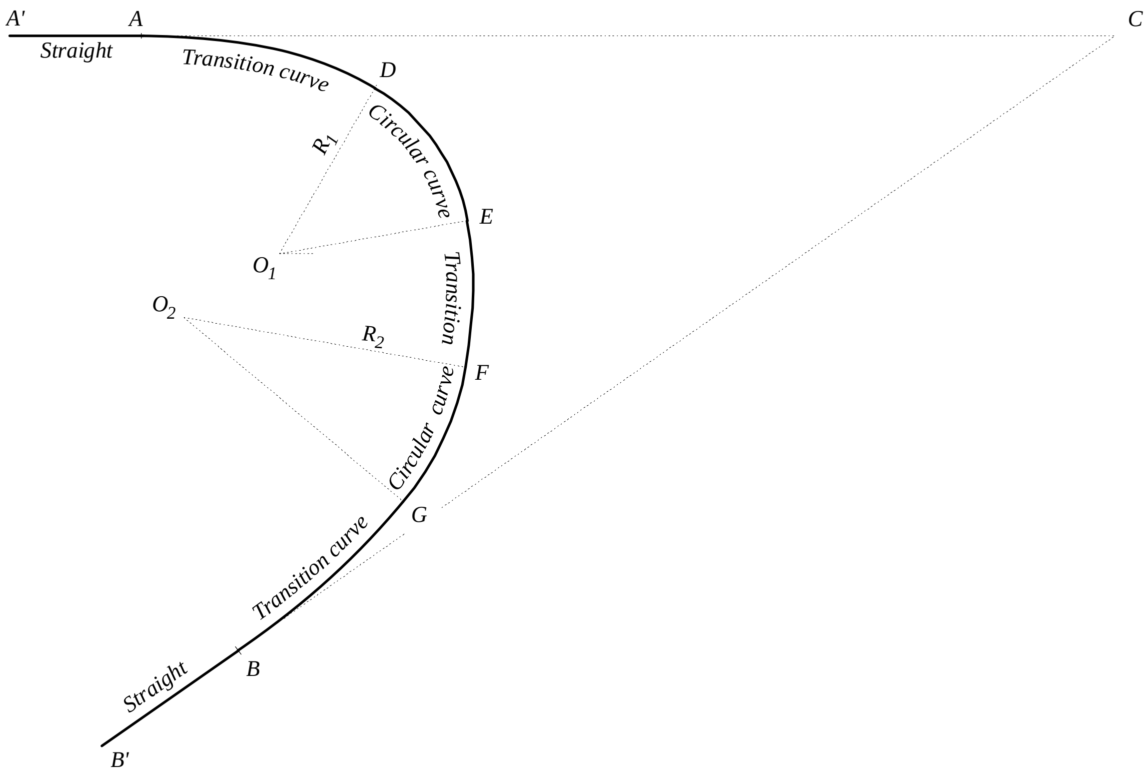


Figure 1.5

In Figure 1.5, a combined curve *ADEFGB* joins the straights *A'A* and *BB'* which intersect at *C*.

2. GEOMETRY OF CIRCULAR CURVES

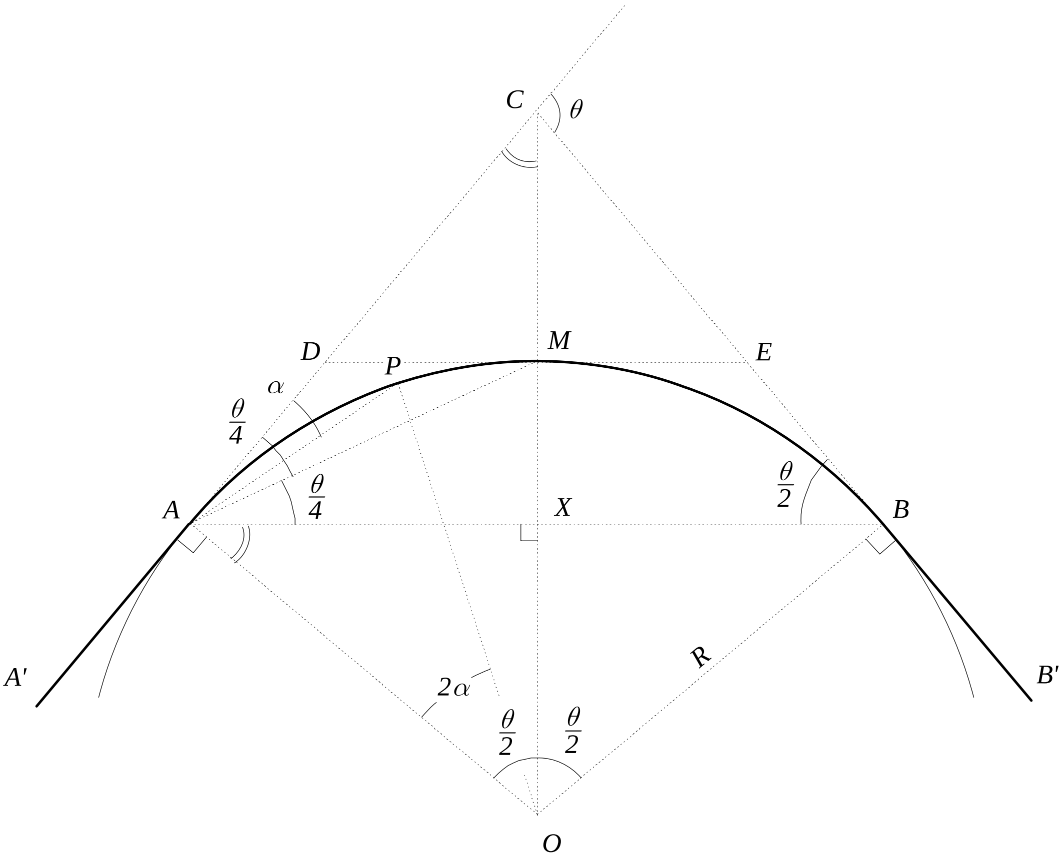


Figure 2.1

Figure 2.1 shows a circular curve *APMB* of radius *R*, centre *O*, joining two straights *A'A* and *B'B* which intersect at *C*. The angle of intersection is θ. *A* and *B* are tangent points and the radials *OA* and *OB* intersect the straights at right angles. *M* is the mid-point of the circular arc *AB* and the mid-point of the line *DE*. *DE* and the chord *AB* are parallel and *X* is the mid-point of the chord *AB*. The chord *AB* is perpendicular to the straight line *OXMC*.

In the quadrilateral *OACB*, the angles *A* and *B* both equal 90° and $C = 180^\circ - \theta$, therefore $O = \theta$. *OACB* is known as a cyclic quadrilateral, (a quadrilateral inscribed within a circle whose opposite angles add to 180°). Due to symmetry $AOC = BOC = \theta/2$ and $ACO = OAB = 90^\circ - \theta/2$. Hence, in the right-angle triangles *AXC* and *BXC*, $CAB = CBA = \theta/2$. Therefore, the angle between the tangent *AC* and the chord *AB* is half the angle subtended at the centre of the circle by the chord *AB*. This is a general property of chords and tangents to circles.

The following formulae may be deduced from Figure 2.1.

Tangent length *AC* $$T = R\tan\frac{\theta}{2} \tag{2.1}$$

Arc length *AB* $$A = R\theta \tag{2.2}$$

Chord length *AB* $$C = 2R\sin\frac{\theta}{2} \tag{2.3}$$

Mid ordinate distance *XM* $$M = R\left(1-\cos\frac{\theta}{2}\right) \tag{2.4}$$

Secant distance *MC* $$S = R\left(\sec\frac{\theta}{2}-1\right) \tag{2.5}$$

3. GEOMETRY OF COMPOUND CIRCULAR CURVES

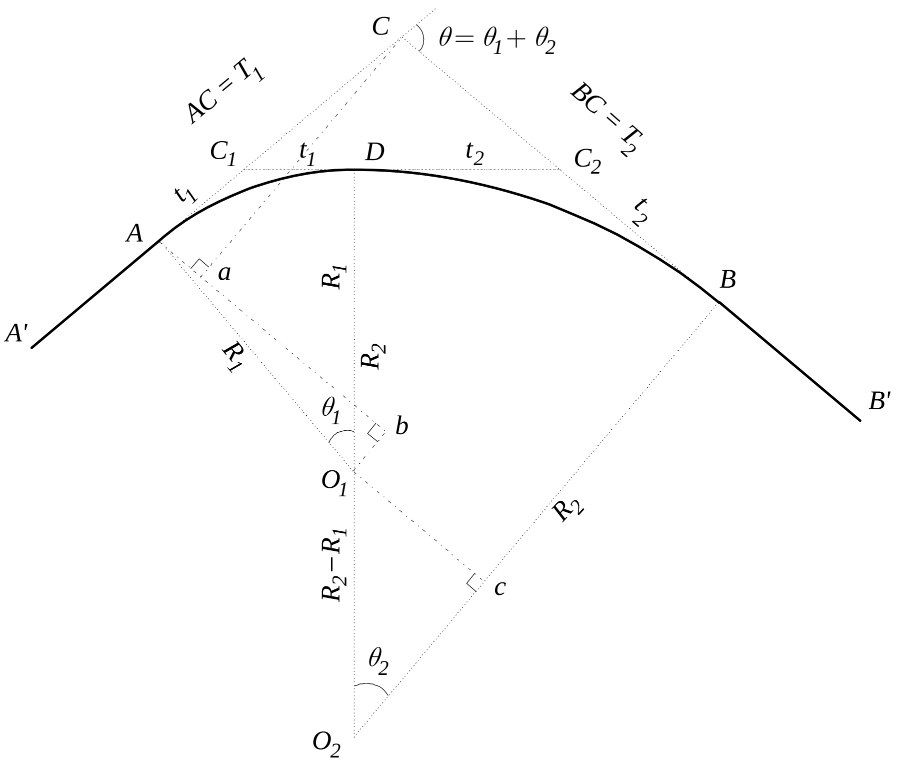


Figure 3.1

In Figure 3.1, a compound circular curve *ADB* joins two straights *A'A* and *BB'* which intersect at *C*. A and *B* are tangent points to circular arcs of radii R_1 and R_2 respectively, whose centres are O_1 and O_2. *D* is a common tangent point and the line C_1C_2 is tangential to both circular curves and perpendicular to the line DO_1O_2. $T_1 = AC$, $T_2 = BC$ are tangent lengths and $A_1 =$ arc AD, $A_2 =$ arc DB are arc lengths of the circular curves.

There are nine elements of a compound circular curve, $\theta, \theta_1, \theta_2$, R_1, R_2, T_1, T_2, A_1 and A_2 and the following formulae linking these elements may be deduced from Figure 3.1.

$$\theta = \theta_1 + \theta_2 \tag{3.1}$$

$$A_1 = R_1\theta_1 \tag{3.2}$$

$$A_2 = R_2\theta_2 \tag{3.3}$$

In the polygon $O_2O_1ACBO_2$ the algebraic sum of the projections of the sides onto any one side must be zero. In Figure 3.1, considering the projections of the sides onto the radius O_2B we may write $Ca = BO_2 - O_2c - O_1b$ or

$$\begin{aligned} T_1 \sin\theta &= R_2 - \left(R_2 - R_1\right)\cos\theta_2 - R_1\cos\theta \\ &= R_2 - R_2\cos\theta_2 + R_1\cos\theta_2 - R_1\cos\theta \\ &= R_2\left(1-\cos\theta_2\right) + R_1\left(\cos\theta_2 - \cos\theta\right) \\ &= R_2\left(1-\cos\theta_2\right) + R_1\left(1-\cos\theta - \left(1-\cos\theta_2\right)\right) \end{aligned}$$

which simplifies to

$$T_1 \sin\theta = R_1\left(1-\cos\theta\right) + \left(R_2 - R_1\right)\left(1-\cos\theta_2\right) \tag{3.4}$$

Similarly, projecting onto the radius O_1A gives

$$T_2 \sin\theta = R_2\left(1-\cos\theta\right) - \left(R_2 - R_1\right)\left(1-\cos\theta_1\right) \tag{3.5}$$

Expressions for the tangent distances T_1 and T_2 can be obtained by considering the tangent distances t_1 and t_2

$$t_1 = R_1 \tan\frac{\theta_1}{2} \tag{3.6}$$

$$t_2 = R_2 \tan\frac{\theta_2}{2} \tag{3.7}$$

and using the sine rule in triangle C_1CC_2

$$CC_1 = T_1 - t_1 = \left(t_1 + t_2\right)\frac{\sin\theta_2}{\sin\theta}$$

giving

$$T_1 = t_1 + \left(t_1 + t_2\right)\frac{\sin\theta_2}{\sin\theta} \tag{3.8}$$

and similarly

$$T_2 = t_2 + \left(t_1 + t_2\right)\frac{\sin\theta_1}{\sin\theta} \tag{3.9}$$

In some compound curve computations, the equations above are not convenient for solving unknowns. In such circumstances an "equivalent" circle of radius *R*, which is tangential to all three lines *AA′*, *BB′* and C_1C_2 may be introduced and equations developed.

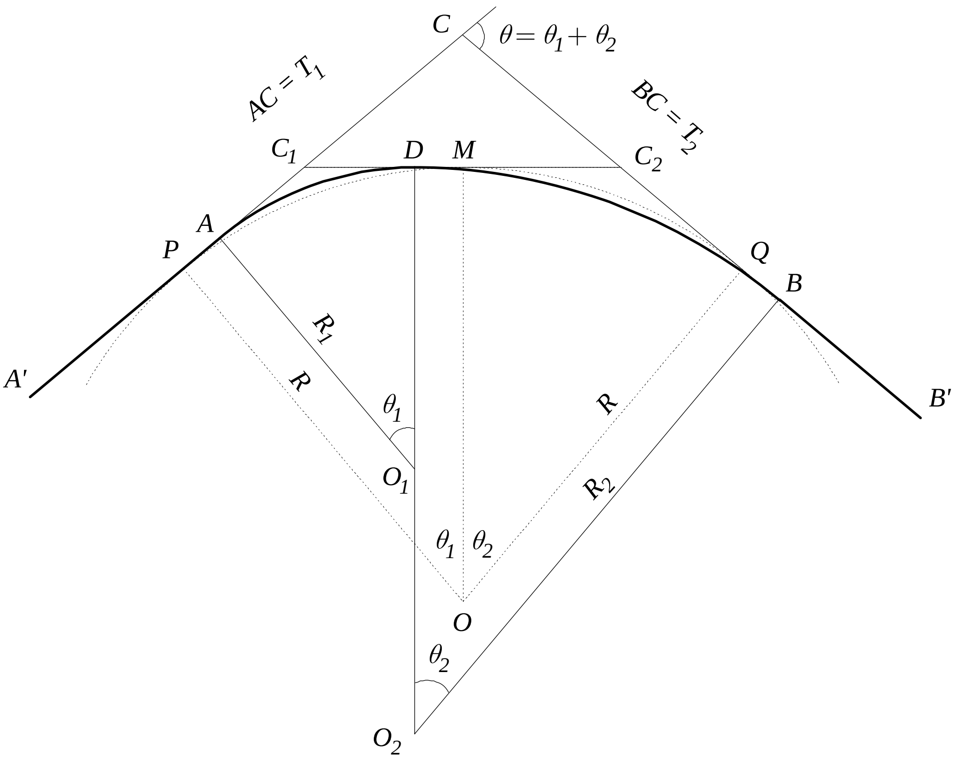


Figure 3.2

In Figure 3.2, the circular curve (dotted) *PMQ* of radius *R*, centred at *O*, is tangential to the two straights *AA′* and *BB′* and the line C_1C_2. The tangent points are *P*, *M* and *Q*. Using the formula for tangent length

$$PA = PC_1 - AC_1 = R\tan\frac{\theta_1}{2} - R_1\tan\frac{\theta_1}{2} = \left(R - R_1\right)\tan\frac{\theta_1}{2}$$

Similarly

$$QB = BC_1 - QC_1 = R_2\tan\frac{\theta_2}{2} - R\tan\frac{\theta_2}{2} = \left(R_2 - R\right)\tan\frac{\theta_2}{2}$$

Now, since $PA = DM$ and $QB = DM$ then $PA = QB$ hence

$$DM = \left(R - R_1\right)\tan\frac{\theta_1}{2} = \left(R_2 - R\right)\tan\frac{\theta_2}{2} \qquad (3.10)$$

Re-arranging the equation gives the radius of the equivalent circular curve

$$R = \frac{R_1\tan\frac{\theta_1}{2} + R_2\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2} + \tan\frac{\theta_1}{2}} \qquad (3.11)$$

Also

$$AC = CP - DM \tag{3.12}$$

$$BC = CQ + DM \tag{3.13}$$

where
$$CP = CQ = R\tan\frac{\theta}{2} \tag{3.14}$$

Example: Given: $\theta = 75^\circ$, $\theta_1 = 30^\circ$, $\theta_2 = 45^\circ$, AC = 180.000 m and BC = 215.000 m.
Compute: R_1 and R_2.

Using equations (3.12), (3.13) and (3.14)

$$180 = CP - DM$$
$$215 = CP + DM$$

From which we obtain $2(CP) = 395$ thus CP = 197.500 m and DM = 17.500 m.

Since CP is now known and $\theta = 75^\circ$, then from (3.14) R = 257.387 m.

Since DM is now known, then from (3.10) R_1 = 192.076 m and R_2 = 299.636 m

4. GEOMETRY OF REVERSE CIRCULAR CURVES

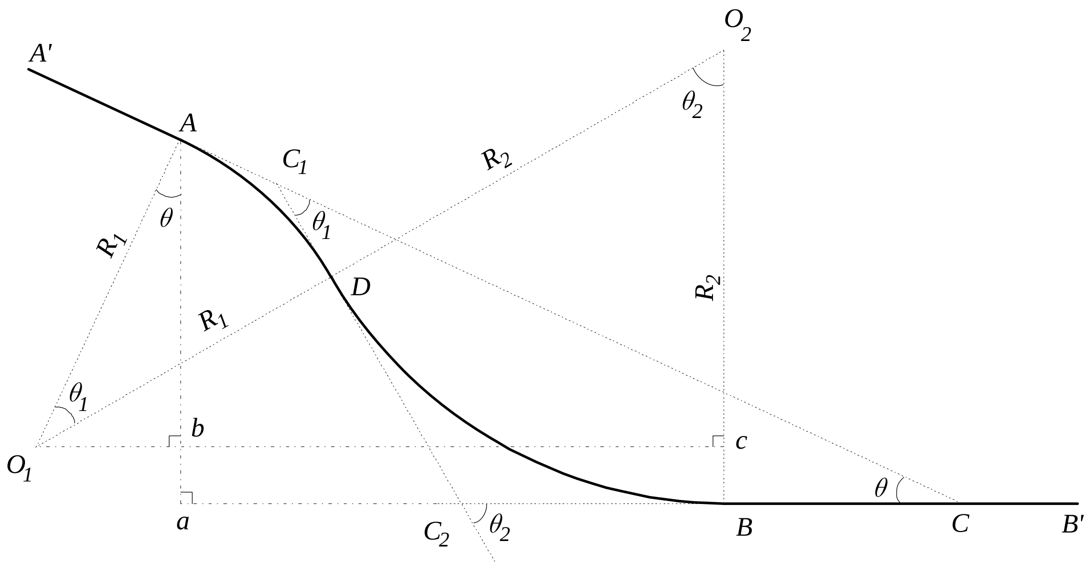


Figure 4.1

In Figure 4.1, a reverse circular curve *ADB* joins two straights *A'A* and *BB'*. A and *B* are tangent points to circular arcs of radii R_1 and R_2 respectively. *D* is a common tangent point. C_1 and C_2 are intersection points and the line C_1C_2 is perpendicular to the line between the centres O_1DO_2. *C* is an intersection point created by extending *AA'* to intersect *BB'*.

Similarly to compound circular curves, there are nine elements of a reverse circular curve, θ, θ_1, θ_2, R_1, R_2, T_1, T_2, A_1 and A_2 and the following formulae linking these elements may be deduced from Figure 4.1.

From triangle CC_1C_2, $\theta + \theta_1 + \left(180^\circ - \theta_2\right) = 180^\circ$ from which we obtain $\theta = \theta_2 - \theta_1$. For other reverse curves, it may be that $\theta = \theta_1 - \theta_2$ but in <u>all</u> cases, θ is the positive difference between θ_1 and θ_2 or the magnitude of the difference

$$\theta = \left|\theta_1 - \theta_2\right| \tag{4.1}$$

As before

$$A_1 = R_1\theta_1 \tag{4.2}$$

$$A_2 = R_2\theta_2 \tag{4.3}$$

As with the compound circular curve, the algebraic sum of projections of certain lines can be used to derive a formula linking the elements of the reverse curve.

Considering Figure 4.1, we may write $Aa = Ab - cO_2 + O_2B$ or

$$\begin{aligned} AC\sin\theta &= R_1\cos\theta - \left(R_1 + R_2\right)\cos\theta_2 + R_2 \\ &= R_1\cos\theta - R_1\cos\theta_2 - R_2\cos\theta_2 + R_2 \\ &= R_1\left(\cos\theta - \cos\theta_2\right) + R_2\left(1 - \cos\theta_2\right) \\ &= R_1\left(1 - \cos\theta_2 - \left(1 - \cos\theta\right)\right) + R_2\left(1 - \cos\theta_2\right) \end{aligned}$$

which simplifies to

$$AC\sin\theta = \left(R_1 + R_2\right)\left(1 - \cos\theta_2\right) - R_1\left(1 - \cos\theta\right) \tag{4.4}$$

Using a similar technique

$$BC\sin\theta = \left(R_1 + R_2\right)\left(1 - \cos\theta_1\right) - R_2\left(1 - \cos\theta\right) \tag{4.5}$$

5. GEOMETRY OF TRANSITION CURVES

A transition curve is a curve whose *curvature* κ (kappa) varies uniformly with respect to its length and allows a gradual change from one radius to another. Or from a straight line to a circular curve, since a straight line is merely a curve of infinite radius. The concept of curvature and its reciprocal, *radius of curvature* ρ (rho), is discussed below.

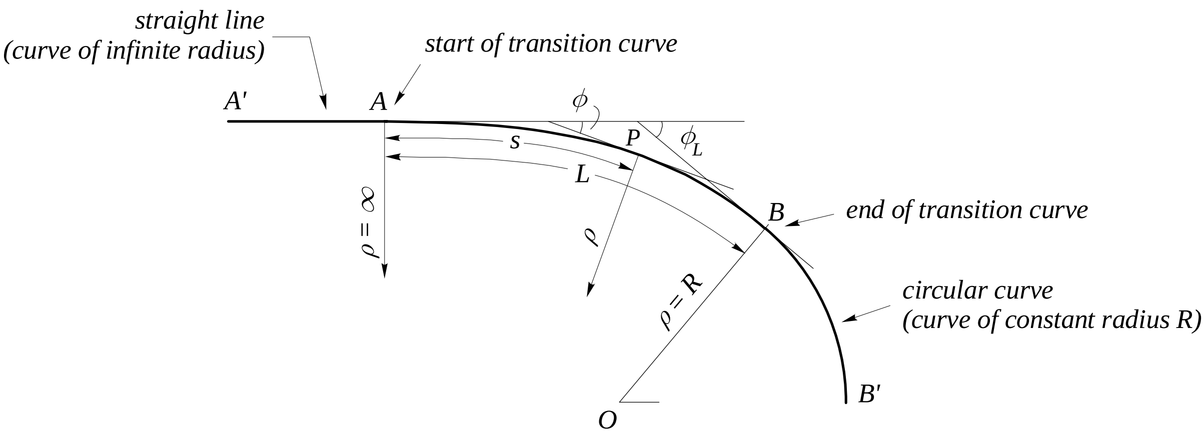


Figure 5.1

Figure 5.1 shows a transition curve linking the straight *A′A* with the circular curve *BB′*. *P* is a point on the transition curve at some distance *s* (arc length) from *A*. The total length of the transition curve is *L*. At *P*, the transition curve has a *radius of curvature* ρ , at *A* $\rho = \infty$ (infinity) and at *B*, the beginning of the circular curve, $\rho = R$. The tangent to the transition curve at *P* intersects the extension of *A′A* at an angle of ϕ , known as the *tangential angle*. ϕ has a value of zero at *A* (the beginning of the curve) and a maximum value of ϕ_1 at *B* (the end of the curve). In any transition curve, the change in ϕ is proportional to the change in *s*.

5.1 Curvature κ and Radius of Curvature ρ

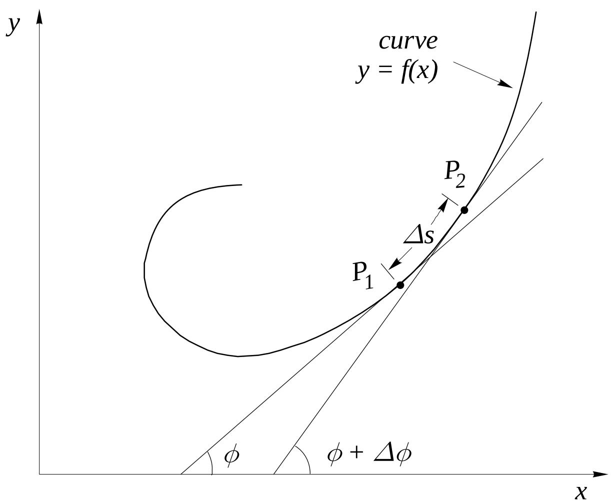


Figure 5.2

Figure 5.2 shows a curve $y = f(x)$ and two points on the curve P_1 and P_2 whose tangents cut the *x*-axis at angles ϕ and $\phi + \Delta\phi$. The distance along the curve between P_1 and P_2 is Δs. The *curvature* κ of a curve $y = f(x)$ at any point *P* is the rate of change of direction of the curve, (i.e., the change in the inclination of the tangent) with respect to the arc length *s*. The curvature is defined as

$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds} \tag{5.1}$$

The *radius of curvature* ρ is defined as the inverse of the curvature

$$\rho = \frac{1}{\kappa} \quad \text{where } \kappa \neq 0 \tag{5.2}$$

The radius of curvature can be thought of as the radius of a circle, which "best fits" the curve at that point. A circle has a constant radius of curvature (and hence a constant curvature) and a straight line has an infinite radius of curvature, or a curvature of zero.

5.2 The equation of the transition curve

A transition curve is defined as having a constant rate of change of curvature with respect to the arc length, i.e., if ϕ is the tangential angle and *s* is the arc length, then

$$\frac{d\kappa}{ds} = \frac{d^2\phi}{ds^2} = K \quad \text{where } K \text{ is a constant} \tag{5.3}$$

Consider the case of a transition curve joining a straight and a circular curve of constant radius *R* as in Figure 5.1. Integrating (5.3) gives

$$\frac{d\phi}{ds} = \int K \; ds = Ks + K_1$$

K_1 is a constant of integration which can be determined by considering the following; at *A*, the start of the curve, $s = 0$ and the curvature is also zero, i.e., $d\phi / ds = 0$, hence $K_1 = 0$ and

$$\frac{d\phi}{ds} = Ks \tag{5.4}$$

Integrating again

$$\phi = \int Ks \; ds = \frac{Ks^2}{2} + K_2$$

Again, K_2 is a constant of integration which can be determined, since at the start of the curve, $s = 0$ and $\phi = 0$, hence $K_2 = 0$ and

$$\phi = \frac{Ks^2}{2} \tag{5.5}$$

Equation (5.5) is the fundamental equation of the transition curve or *clothoid*, one of a family of mathematical curves known as *spirals*. The clothoid is also known as Euler's spiral or Cornu's spiral.

Equation (5.5) may be written as

$$s = C\sqrt{\phi} \tag{5.6}$$

where $C = \sqrt{\frac{2}{K}}$. If *L* is the total length of the clothoid, then when *s* = *L*, i.e., at the end of the curve and the beginning of the circular curve, the curvature $\kappa = d\phi / ds = 1/\rho = 1/R$ and from (5.4) $d\phi / ds = Ks = KL$. Hence equating the derivatives gives the constants $K = 1/(LR)$ and the equation of the clothoid becomes

$$\phi = \frac{s^2}{2LR} \tag{5.7}$$

or

$$s = \sqrt{2LR\phi} \tag{5.8}$$

Note: Since the curvature $\kappa = d\phi / ds = Ks = s/(LR) = 1/\rho$, where ρ is the radius of curvature corresponding to the arc s then

$$s\rho = LR = \text{ constant} \tag{5.9}$$

This is an important and useful property of the clothoid.

When $s = L$ (i.e., at the end of the transition curve and the beginning of the circular curve) the total tangential angle ϕ_L is determined from (5.7) as

$$\phi_L = \frac{L}{2R} \tag{5.10}$$

5.3 Rectangular coordinates of the clothoid transition curve

The formulae above are not suitable for setting out clothoid transition curves in the field. Instead, rectangular coordinates of points on the curve will be more useful.

In Figure 5.3, P is a point on the clothoid, at a distance s from the start of the curve and the tangent to P cuts the x-axis at an angle ϕ. The x-y rectangular coordinate system has an origin at A, the start of the transition curve. The x-axis is the extension of the line $A'A$, i.e., the tangent to the curve at A; the x-coordinate of P is the distance along the tangent and the y-coordinate is the perpendicular offset from the tangent. A small arc length Δs has components Δx and Δy, and in the limit become infinitesimal changes ds, dx and dy shown in the enlargement to the right.

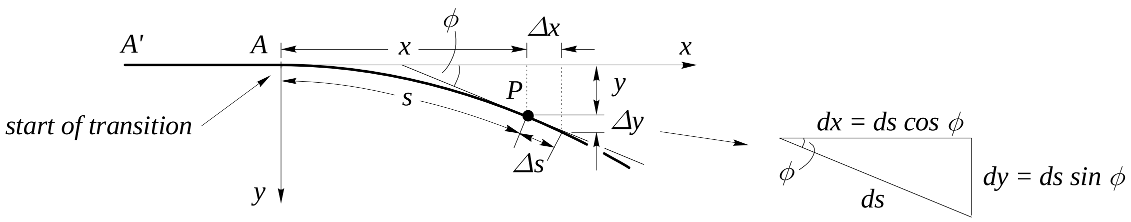


Figure 5.3

To express the equation of the clothoid in rectangular coordinates we make use of the differential relationships shown in Figure 5.3

$$\begin{aligned} dx &= ds\ \cos\phi \\ dy &= ds\ \sin\phi \end{aligned} \tag{5.11}$$

Differentiating (5.7)

$$d\phi = \frac{s}{LR} ds$$

Substituting for s using (5.8) and re-arranging gives

$$ds = \sqrt{\frac{LR}{2}} \frac{d\phi}{\sqrt{\phi}}$$

Substituting for *ds* in equations (5.11) and integrating gives

$$x = \sqrt{\frac{LR}{2}} \int_0^\phi \frac{\cos\phi}{\sqrt{\phi}} d\phi$$

$$y = \sqrt{\frac{LR}{2}} \int_0^\phi \frac{\sin\phi}{\sqrt{\phi}} d\phi$$

These integrals, known as Fresnel integrals cannot be expressed in terms of elementary functions. Instead, $\cos\phi$ and $\sin\phi$ are expanded into series and the integration performed term by term with the result expressed as a truncated series, assuming successive terms become smaller and smaller. Then

$$x = \sqrt{\frac{LR}{2}} \int_0^\phi \phi^{-1/2} \left\{1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \frac{\phi^8}{8!} - \cdots \right\} d\phi$$

$$y = \sqrt{\frac{LR}{2}} \int_0^\phi \phi^{-1/2} \left\{\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \frac{\phi^9}{9!} - \cdots \right\} d\phi$$

Performing the integrations and simplifying gives the series expansion for the clothoid in terms of the tangential angle ϕ

$$x = \sqrt{2LR}\,\sqrt{\phi} \left\{1 - \frac{\phi^2}{(5)2!} + \frac{\phi^4}{(9)4!} - \frac{\phi^6}{(13)6!} + \frac{\phi^8}{(17)8!} - \cdots \right\} \tag{5.12}$$

$$y = \sqrt{2LR}\,\sqrt{\phi} \left\{\frac{\phi}{3} - \frac{\phi^3}{(7)3!} + \frac{\phi^5}{(11)5!} - \frac{\phi^7}{(15)7!} + \frac{\phi^9}{(19)9!} - \cdots \right\} \tag{5.13}$$

Substituting for ϕ from equation (5.7) gives the series expansion for the clothoid in terms of curve length *s*

$$x = s - \frac{s^5}{\left(5\cdot 2^2\right)2!\left(LR\right)^2} + \frac{s^9}{\left(9\cdot 2^4\right)4!\left(LR\right)^4} - \frac{s^{13}}{\left(13\cdot 2^6\right)6!\left(LR\right)^6} + \frac{s^{17}}{\left(17\cdot 2^8\right)8!\left(LR\right)^8} - \cdots \tag{5.14}$$

$$y = \frac{s^3}{\left(3\cdot 2^1\right)LR} - \frac{s^7}{\left(7\cdot 2^3\right)3!\left(LR\right)^3} + \frac{s^{11}}{\left(11\cdot 2^5\right)5!\left(LR\right)^5} - \frac{s^{15}}{\left(15\cdot 2^7\right)7!\left(LR\right)^7} + \frac{s^{19}}{\left(19\cdot 2^9\right)9!\left(LR\right)^9} - \cdots \tag{5.15}$$

Note that $\left(5\cdot 2^2\right) = 5\times 2^2$. Equations (5.14) and (5.15) can be re-arranged into a power series in $\frac{s^2}{LR}$

$$x = s\left\{1 - \frac{1}{5\cdot 2^2\cdot 2!}\left(\frac{s^2}{LR}\right)^2 + \frac{1}{9\cdot 2^4\cdot 4!}\left(\frac{s^2}{LR}\right)^4 - \frac{1}{13\cdot 2^6\cdot 6!}\left(\frac{s^2}{LR}\right)^6 + \frac{1}{17\cdot 2^8\cdot 8!}\left(\frac{s^2}{LR}\right)^8 - \cdots \right\} \tag{5.16}$$

$$y = \frac{s^3}{6LR}\left\{1 - \frac{6}{7\cdot 2^3\cdot 3!}\left(\frac{s^2}{LR}\right)^2 + \frac{6}{11\cdot 2^5\cdot 5!}\left(\frac{s^2}{LR}\right)^4 - \frac{6}{15\cdot 2^7\cdot 7!}\left(\frac{s^2}{LR}\right)^6 + \frac{6}{19\cdot 2^9\cdot 9!}\left(\frac{s^2}{LR}\right)^8 - \cdots \right\} \tag{5.17}$$

The maximum values of *x* and *y* are reached when $s = L$, i.e., at the end of the transition curve. Substituting $s = L$ into equations (5.14) (5.15) gives

$$x_{\max} = L - \frac{L^3}{40R^2} + \frac{L^5}{3456R^4} - \cdots \tag{5.18}$$

$$y_{\max} = \frac{L^2}{6R} - \frac{L^4}{336R^3} + \frac{L^6}{42240R^5} - \cdots \tag{5.19}$$

5.4 Offsets from the tangent to the clothoid transition curve

For setting out purposes, it may be desirable to compute the *y*-coordinate (the offset from the tangent) given the *x*-coordinate (distance along the tangent). To express *y* as a function of *x*, we first obtain *s* in terms of *x* by "reversing" the series in *x* in equation (5.14) using *Lagrange's Theorem*[1]

Given

$$s = x + wF(s) \quad \text{or} \quad x = s - wF(s) \tag{5.20}$$

then

$$\begin{aligned} f(s) = f(x) + wF(x)\,f'(x) &+ \frac{w^2}{2!}\frac{d}{dx}\left[\{F(x)\}^2 f'(x)\right] \\ &+ \frac{w^3}{3!}\frac{d^2}{dx^2}\left[\{F(x)\}^3 f'(x)\right] \\ &+ \cdots \\ &+ \frac{w^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left[\{F(x)\}^n f'(x)\right] \end{aligned} \tag{5.21}$$

where $f(s)$ is a function of *s*, $f(x)$ and $F(x)$ are functions of *x*, $f'(x)$ is the derivative of $f(x)$ and *w* is a constant.

In our case $w = 1$ and we choose $f(s) = s$ so that $f(x) = x$ and $f'(x) = 1$ giving

$$s = x + F(x) + \frac{1}{2!}\frac{d}{dx}\{F(x)\}^2 + \cdots + \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}\{F(x)\}^n \tag{5.22}$$

Now $F(s)$ consists of all terms on the right-hand side of (5.14), except the 1st term, noting the change of sign to accord with $x = s - F(s)$ in equation (5.20)

$$F(s) = \frac{s^5}{40(LR)^2} - \frac{s^9}{3456(LR)^4} + \cdots$$

hence $F(x)$ is the same series with *x* replacing *s*

$$F(x) = \frac{x^5}{40(LR)^2} - \frac{x^9}{3456(LR)^4} + \cdots$$

This is the 2nd term in equation (5.22). The 3rd term is obtained as follows

$$\begin{aligned} \{F(x)\}^2 &= \frac{x^{10}}{1600(LR)^4} - \frac{2x^{14}}{138240(LR)^6} + \frac{x^{18}}{11943936(LR)^8} + \cdots \\ \frac{1}{2}\frac{d}{dx}\{F(x)\}^2 &= \frac{1}{2}\frac{10x^9}{1600(LR)^4} + \cdots \\ &= \frac{x^9}{320(LR)^4} + \cdots \end{aligned}$$

---

[1] Reversion of a series can be achieved by using *Lagrange's Theorem.* A proof of this theorem can be found in *Formulas and Theorems in Pure Mathematics* by George S. Carr (2nd ed, Chelsea Pub. Co., New York, 1970). An application of *Lagrange's Theorem* can be found in *Geodesy and Map Projections,* by G.B. Lauf (TAFE Publications, Collingwood, Aust., 1983), where it is used to derive a series expression for the "foot-point" latitude used in conversion of latitudes and longitudes (geodetic coordinates) to Universal Transverse Mercator projection coordinates.

The series in equation (5.22) becomes

$$\begin{aligned} s &= x + \left(\frac{x^5}{40(LR)^2} + \frac{x^9}{3456(LR)^4} + \cdots \right) + \left(\frac{x^9}{320(LR)^4} \cdots \right) + \cdots \\ &= x + \frac{x^5}{40(LR)^2} + \frac{49x^9}{17280(LR)^4} + \cdots \end{aligned} \tag{5.23}$$

Substituting this series for s into equation (5.15) gives the series for y in terms of x

$$y = \frac{x^3}{6LR} + \frac{x^7}{105(LR)^3} + \frac{293x^{11}}{237600(LR)^5} + \frac{55397x^{15}}{269568000(LR)^7} - \frac{131021x^{19}}{7763558400(LR)^9} - \cdots \tag{5.24}$$

5.5 Polar coordinates of the clothoid transition curve

For "setting-out" the clothoid, it may be desirable to determine the polar coordinates of P on the curve.

Figure 5.4 shows P having x,y rectangular coordinates. The polar coordinates of P are c, the chord distance and α the "deflection angle" from the tangent (the x-axis).

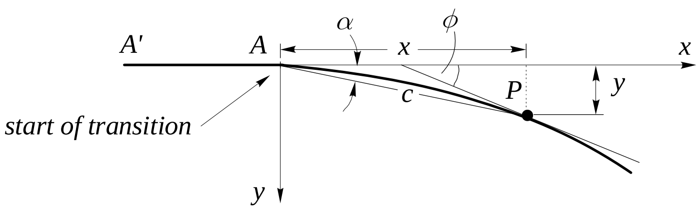


Figure 5.4

It can be seen from Figure 5.4 that

$$\tan\alpha = \frac{y}{x} \tag{5.25}$$

and that $c = \sqrt{x^2 + y^2}$ or preferably

$$c = \frac{x}{\cos\alpha} \tag{5.26}$$

In practical problems, c and α are calculated from the x and y coordinates computed from the series equations above.

5.6 The *shift S* of a transition curve

To insert a transition curve between a straight and a circular curve it is necessary to shift the circular curve away from the straight by an amount known as the *shift*. Similarly in order to insert a transition curve between two circular curves forming a compound curve it is necessary to move the circular curve with the smaller radius inwards, or the circular curve with the larger radius outwards.

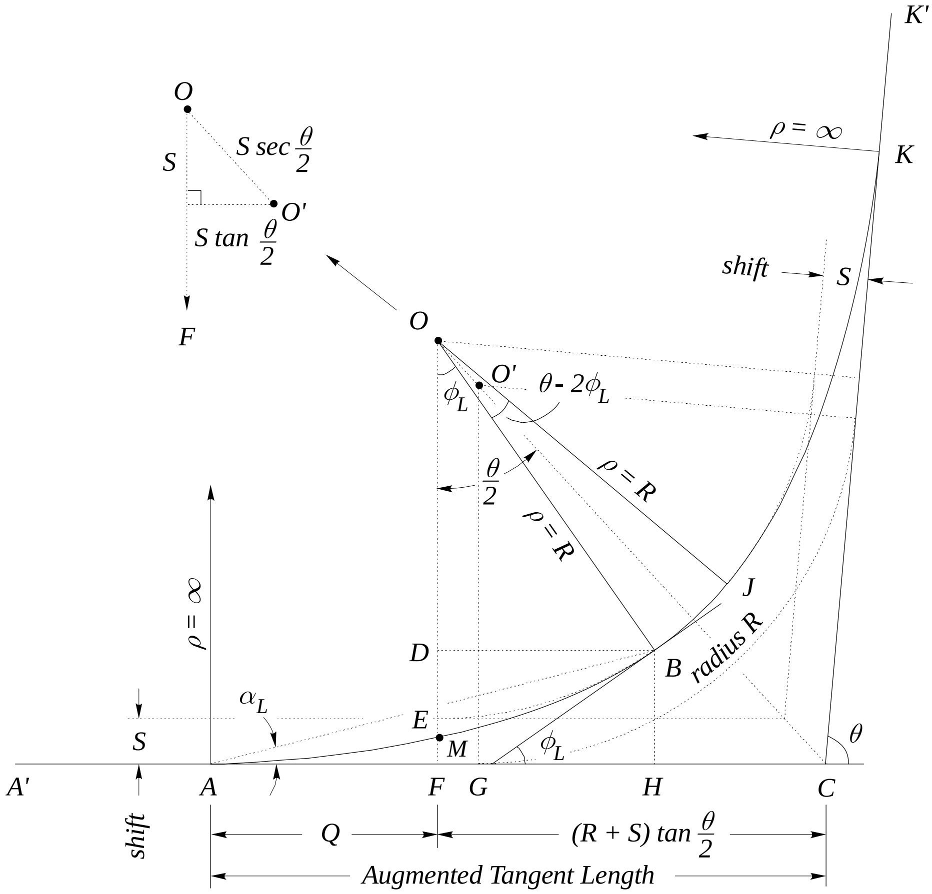


Figure 5.5

In Figure 5.5, straights *A'A* and *KK'* intersect at *C*. The intersection angle is θ. A circular curve of radius *R*, centred at *O'* was originally used to join the two straights, but has been shifted to *O* to allow for the introduction of two transition curves *AB* and *JK*, both of length *L*. *O'* has been shifted to *O* a distance $S\sec\dfrac{\theta}{2}$ where *S* is the *shift*, the perpendicular distance *EF*.

From Figure 5.5

$$\begin{aligned} S &= BH - DE \\ &= y_{\max} - R\left(1 - \cos\phi_L\right) \end{aligned}$$

where $y_{\max}$ is the maximum offset from the tangent given by equation (5.19) and $\phi_L = \dfrac{L}{2R}$ is the maximum tangential angle (see equation 5.10). Using the expansion $\cos\phi = 1 - \dfrac{\phi^2}{2!} + \dfrac{\phi^4}{4!} - \dfrac{\phi^6}{6!} + \cdots$ and substituting for ϕ_L and $y_{\max}$ gives

$$S=\left(\frac{L^2}{6R}-\frac{L^4}{336R^3}+\frac{L^6}{42240R^5}-\cdots\right)-R\left(1-\left(1-\frac{L^2}{8R^2}+\frac{L^4}{384R^4}-\frac{L^6}{46080R^6}+\cdots\right)\right)$$
$$=\left(\frac{L^2}{6R}-\frac{L^4}{336R^3}+\frac{L^6}{42240R^5}-\cdots\right)-\left(\frac{L^2}{8R}-\frac{L^4}{384R^3}+\frac{L^6}{46080R^5}-\cdots\right)$$

which simplifies to

$$S=\frac{L^2}{24R}-\frac{L^4}{2688R^3}+\frac{L^6}{506880R^5}-\cdots \tag{5.27}$$

For many practical applications the shift is approximated by

$$S\simeq\frac{L^2}{24R} \tag{5.28}$$

5.7 The Augmented Tangent Length of a transition curve

From Figure 5.5, the *Augmented Tangent Length* is the distance *AC* where

$$AC = Q + FC \tag{5.30}$$

and

$$FC = FG + GC$$

G is the tangent point of the original circular curve or radius *R* joining the two straights, $GC = R\tan\theta/2$ and $FG = S\tan\theta/2$, hence the Augmented Tangent Length is

$$AC=Q+\left(R+S\right)\tan\frac{\theta}{2} \tag{5.31}$$

From Figure 5.5

$$Q = AH - DB$$
$$= x_{\max} - R\sin\phi_L$$

where $x_{\max}$ is the maximum distances along the tangent given by equation (5.18) and $\phi_L=\dfrac{L}{2R}$ is the maximum tangential angle (see equation 5.10). Using the expansion $\sin\phi=\phi-\dfrac{\phi^3}{3!}+\dfrac{\phi^5}{5!}-\dfrac{\phi^7}{7!}+\cdots$ and substituting for ϕ_L and $x_{\max}$ gives

$$Q=\left(L-\frac{L^3}{40R^2}+\frac{L^5}{3456R^4}-\cdots\right)-R\left(\frac{L}{2R}-\frac{L^3}{48R^3}+\frac{L^5}{3840R^5}-\cdots\right)$$
$$=\left(L-\frac{L^3}{40R^2}+\frac{L^5}{3456R^4}-\cdots\right)-\left(\frac{L}{2}-\frac{L^3}{48R^2}+\frac{L^5}{3840R^4}-\cdots\right)$$

which simplifies to

$$Q=\frac{L}{2}-\frac{L^3}{240R^2}+\frac{L^5}{34560R^4}-\cdots \tag{5.32}$$

The Augmented Tangent Length *AC* becomes

$$AC=\left(\frac{L}{2}-\frac{L^3}{240R^2}+\frac{L^5}{34560R^4}-\cdots\right)+\left(R+S\right)\tan\frac{\theta}{2} \tag{5.33}$$

5.8 Clothoid transition curves between circular curves

Two circular curves of radii R_1 and R_2 can be joined by a clothoid transition curve whose curvature varies from $\kappa_1 = 1/\rho_1 = 1/R_1$ to $\kappa_2 = 1/\rho_2 = 1/R_2$. That is, a transition curve tangential to both circular arcs

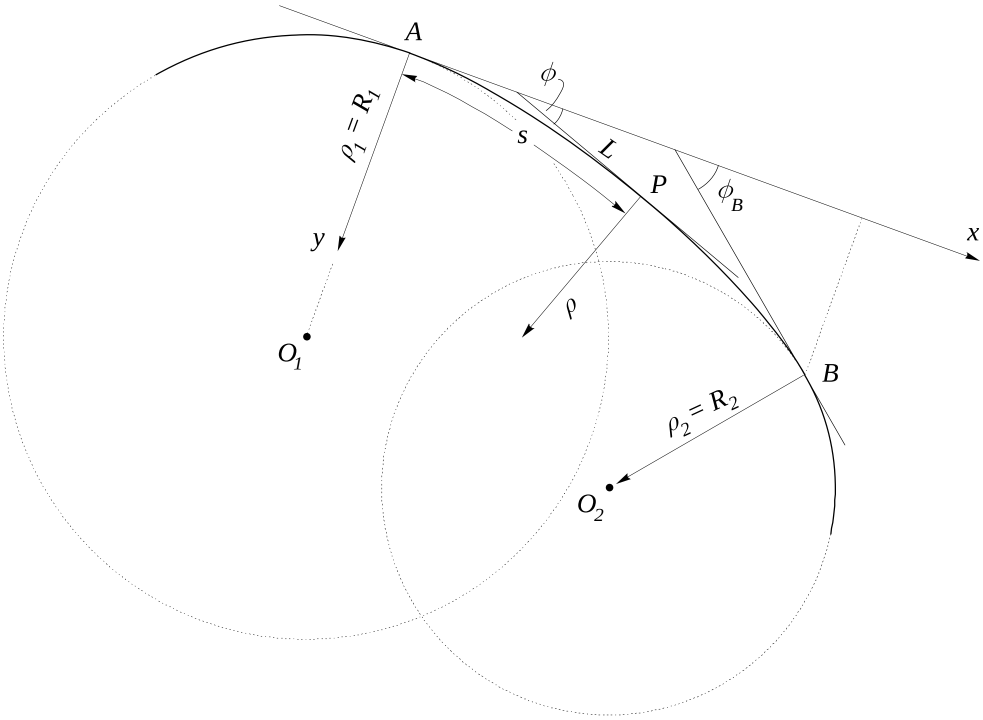


Figure 5.6

Figure 5.6 shows two circular curves of radii R_1 and R_2 centred at O_1 and O_2. A clothoid transition curve *AB* of length *L* joins these two circular curves. The curvature of the clothoid at *A* is $\kappa_1 = 1/\rho_1 = 1/R_1$ and the curvature at *B* is $\kappa_2 = 1/\rho_2 = 1/R_2$ and the clothoid has a constant rate of change of curvature with respect to arc length *s*. Hence, we may link the curvature at *P* with the curvatures at the beginning and end of the curve by

$$\kappa_P = \kappa_1 + \frac{(\kappa_2 - \kappa_1)}{L} s \tag{5.34}$$

The elemental arc length at *P* is

$$ds = \rho \; d\phi = \frac{1}{\kappa_P} \; d\phi \tag{5.35}$$

Substituting equation (5.34) and re-arranging gives

$$d\phi = \left(\kappa_1 + \frac{(\kappa_2 - \kappa_1)}{L} s \right) ds$$

Integrating gives an expression for the tangential angle

$$\phi = \kappa_1 s + \frac{\kappa_2 - \kappa_1}{2L} s^2 + C$$

where *C* is a constant of integration.

Since $\phi = 0$ when $s = 0$ then $C = 0$ and the equation for the tangential angle ϕ becomes

$$\begin{aligned}\phi &= \kappa_1 s + \frac{\kappa_2 - \kappa_1}{2L} s^2 \\ &= \frac{s}{R_1} + \frac{R_1 - R_2}{2LR_1R_2} s^2\end{aligned}$$

and letting

$$A = \frac{R_1 - R_2}{2LR_1R_2} \tag{5.36}$$

gives

$$\phi = \frac{s}{R_1} + As^2 \tag{5.37}$$

Now similarly to before, the elemental distance *ds* has components in the *x* and *y* directions, where the *x*,*y* axes have an origin at *A* with the *x*-axis in the direction of the tangent

$$\begin{aligned}dx &= ds\ \cos\phi \\ dy &= ds\ \sin\phi\end{aligned} \tag{5.38}$$

Substituting equation (5.37) for ϕ in equations (5.38), then expanding using the series expansions for $\cos\phi$ and $\sin\phi$, and then integrating gives

$$x = \int_0^s \left\{ 1 - \frac{1}{2!}\left(As^2 + \frac{s}{R_1}\right)^2 + \frac{1}{4!}\left(As^2 + \frac{s}{R_1}\right)^4 - \frac{1}{6!}\left(As^2 + \frac{s}{R_1}\right)^6 + \cdots \right\} ds \tag{5.39}$$

$$y = \int_0^s \left\{ As^2 + \frac{s}{R_1} - \frac{1}{3!}\left(As^2 + \frac{s}{R_1}\right)^3 + \frac{1}{5!}\left(As^2 + \frac{s}{R_1}\right)^5 - \frac{1}{7!}\left(As^2 + \frac{s}{R_1}\right)^7 + \cdots \right\} ds \tag{5.40}$$

Performing the integrations and simplifying (using the symbolic mathematical package MAPLE) gives

$$\begin{aligned}x = s &- \left(\frac{1}{6R_1^2}\right)s^3 - \left(\frac{A}{4R_1}\right)s^4 + \left(\frac{1}{120R_1^4} - \frac{A^2}{10}\right)s^5 + \left(\frac{A}{36R_1^3}\right)s^6 + \left(\frac{A^2}{28R_1^2} - \frac{1}{5040R_1^6}\right)s^7 \\ &+ \left(\frac{A^3}{48R_1} - \frac{A}{960R_1^5}\right)s^8 + \left(\frac{A^4}{216} - \frac{A^2}{432R_1^4}\right)s^9 - \left(\frac{A^3}{360R_1^3}\right)s^{10} \\ &- \left(\frac{A^4}{528R_1^2}\right)s^{11} - \left(\frac{A^5}{1440R_1}\right)s^{12} - \left(\frac{A^6}{9360}\right)s^{13} + \cdots\end{aligned} \tag{5.41}$$

$$\begin{aligned}y = &\left(\frac{1}{2R_1}\right)s^2 + \left(\frac{A}{3}\right)s^3 - \left(\frac{1}{24R_1^3}\right)s^4 - \left(\frac{A}{10R_1^2}\right)s^5 + \left(\frac{1}{720R_1^5} - \frac{A^2}{12R_1}\right)s^6 + \left(\frac{A}{168R_1^4} - \frac{A^3}{42}\right)s^7 \\ &+ \left(\frac{A^2}{96R_1^3} - \frac{1}{40320R_1^7}\right)s^8 + \left(\frac{A^3}{108R_1^2} - \frac{A}{6480R_1^6}\right)s^9 + \left(\frac{A^4}{240R_1} - \frac{A^2}{2400R_1^5}\right)s^{10} \\ &+ \left(\frac{A^5}{1320} - \frac{A^3}{1584R_1^4}\right)s^{11} - \left(\frac{A^4}{1728R_1^3}\right)s^{12} - \left(\frac{A^5}{3120R_1^2}\right)s^{13} \\ &- \left(\frac{A^6}{10080R_1}\right)s^{14} - \left(\frac{A^7}{75600R_1^2}\right)s^{15} + \cdots\end{aligned} \tag{5.42}$$

5.9 Perpendicular Offsets to a Clothoid Transition Curve

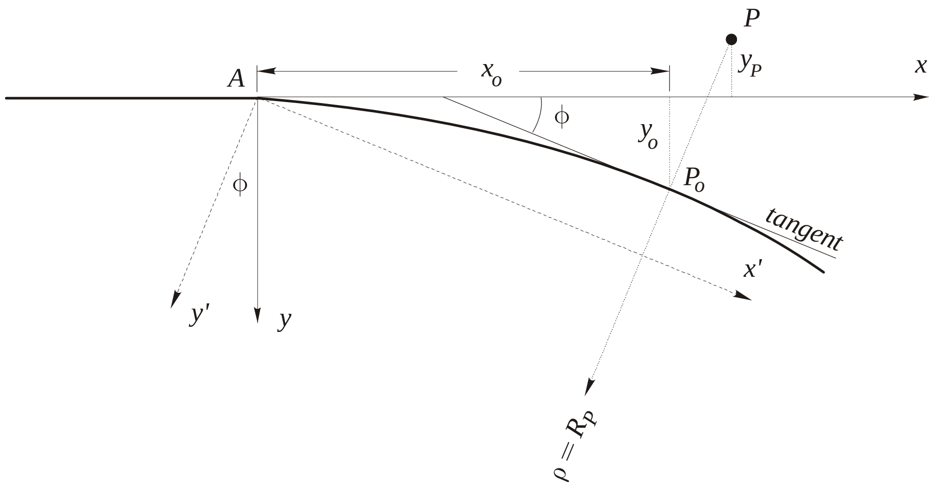


Figure 5.7

Certain purposes may require the computation of perpendicular offsets from points of known coordinates to a clothoid transition curve (defined by *L* and *R*).

Figure 5.7 shows a clothoid transition curve tangential to a straight at *A*. The extension of the straight is the *x*-axis and the *y*-axis is perpendicular to the straight and directed towards the centre of curvature. *P* is a known point (coordinates x_P, y_P) and the perpendicular to the transition curve passing through *P* intersects the curve at P_0 (x_0, y_0). The tangent to the curve at P_0 intersects the *x*-axis at an angle ϕ (the tangential angle). The *x'*-axis is parallel to the tangent at P_0 and the *x'*-*y'* axes are rotated from the *x*-*y* axes by the angle ϕ.

The method of solution is to first determine the tangential angle ϕ, and then compute the distance along the curve between *A* and P_0 using equation (5.8) $s = \sqrt{2LR}\sqrt{\phi}$. Having determined the distance *s*, the *x*-*y* coordinates of P_0 can be computed using equations (5.14) and (5.15) and finally the perpendicular offset P_0-*P* computed from coordinate differences.

To determine the tangential angle ϕ the following formulae and relationships are required.

1. The equations of the *x*-*y* coordinate of a clothoid transition curve given *L* and *R*

$$x = \sqrt{2LR}\sqrt{\phi}\left\{1 - \frac{\phi^2}{(5)2!} + \frac{\phi^4}{(9)4!} - \frac{\phi^6}{(13)6!} + \cdots\right\} \tag{5.12}$$

$$y = \sqrt{2LR}\sqrt{\phi}\left\{\frac{\phi}{3} - \frac{\phi^3}{(7)3!} + \frac{\phi^5}{(11)5!} - \frac{\phi^7}{(15)7!} + \cdots\right\} \tag{5.13}$$

2. The equations for a rotation of coordinate axes

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (5.43)$$

or

$$\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= y\cos\phi - x\sin\phi \end{aligned} \qquad (5.44)$$

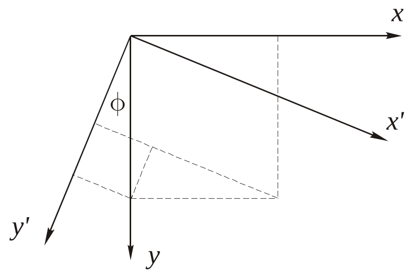


Figure 5.8

3. Sine and Cosine expansions

$$\begin{aligned} \sin\phi &= \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots \\ \cos\phi &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots \end{aligned} \qquad (5.45)$$

The x' coordinate of any point whose x,y coordinates are known is given by (5.44).

$$x' = x\cos\phi + y\sin\phi$$

Substituting the equations for x and y (5.12) and (5.13) and the expansions for sine and cosine (5.45) gives

$$\begin{aligned} x' = \sqrt{2LR}\sqrt{\phi}\left\{1 - \frac{\phi^2}{(5)2!} + \frac{\phi^4}{(9)4!} - \cdots\right\}\left\{1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots\right\} + \\ \sqrt{2LR}\sqrt{\phi}\left\{\frac{\phi}{3} - \frac{\phi^3}{(7)3!} + \frac{\phi^5}{(11)5!} - \cdots\right\}\left\{\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots\right\} \end{aligned}$$

Expanding this equation and then gathering terms gives (1st three terms only) an equation for x' in terms of L, R and the tangential angle ϕ

$$x' = \sqrt{2LR}\,\phi^{1/2} - \frac{4}{15}\sqrt{2LR}\,\phi^{5/2} + \frac{16}{945}\sqrt{2LR}\,\phi^{9/2} \qquad (5.46)$$

The x' coordinate of P is

$$\begin{aligned} x'_P &= x_P\cos\phi + y_P\sin\phi \\ &= x_P\left\{1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots\right\} + y_P\left\{\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots\right\} \end{aligned}$$

Expanding and gathering terms gives

$$x'_P = x_P + y_P\phi - \frac{1}{2}x_P\phi^2 - \frac{1}{6}y_P\phi^3 + \frac{1}{24}x_P\phi^4 + \frac{1}{120}y_P\phi^5 \qquad (5.47)$$

Now, when $x' = x'_P$, the normal to the transition curve will pass through P. Subtracting (5.47) from (5.46) and taking only terms up to the 3rd power gives

$$\sqrt{2LR}\,\phi^{1/2} - \frac{4}{15}\sqrt{2LR}\,\phi^{5/2} - x_P - y_P\phi + \frac{1}{2}x_P\phi^2 + \frac{1}{6}y_P\phi^3 = 0 \qquad (5.48)$$

This equation can be solved for ϕ using Newton's iterative technique. A simplification can be made by using the substitution

$$\alpha = \sqrt{\phi} \qquad (5.49)$$

and equation (5.48) can be written as

$$f(\alpha) = \frac{1}{6} y_P \alpha^6 - \frac{4}{15}\sqrt{2LR}\,\alpha^5 + \frac{1}{2} x_P \alpha^4 - y_P \alpha^2 + \sqrt{2LR}\,\alpha - x_P = 0 \tag{5.50}$$

Solving for $\alpha = \sqrt{\phi}$ using Newton's Iteration

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)} \tag{5.51}$$

where $f'(\alpha)$ is the derivative of $f(\alpha)$ and

$$f'(\alpha) = y_P \alpha^5 - \frac{4}{3}\sqrt{2LR}\,\alpha^4 + 2x_P \alpha^3 - 2y_P \alpha + \sqrt{2LR} \tag{5.52}$$

A starting value for α can be obtained by substituting $\alpha = 0$ into $f(\alpha)$ and $f'(\alpha)$ and using (5.51) to give

$$\alpha_1 = \frac{x_P}{\sqrt{2LR}} \tag{5.53}$$

NOTES

1. A better (computationally) way to calculate numeric values for $f(\alpha)$ and $f'(\alpha)$ is to express the equations in a nested form

$$\begin{aligned} f(\alpha) &= \left(\left(\left(\left(\left(\frac{1}{6} y_P \alpha \right) \alpha - \frac{4}{15}\sqrt{2LR}\;\alpha \right) \alpha + \frac{1}{2} x_P \alpha \right) \alpha - y_P \right) \alpha + \sqrt{2LR} \right) \alpha - x_P \\ f'(\alpha) &= \left(\left(\left((y_P \alpha)\alpha - \frac{4}{3}\sqrt{2LR}\;\alpha \right) \alpha + 2x_P \alpha \right) \alpha - 2y_P \right) \alpha + \sqrt{2LR} \end{aligned} \tag{5.54}$$

2. When using this method to compute perpendicular offsets it should be remembered that the positive directions of the *x* and *y* axes are dictated by the direction of the transition curve. They may be opposite to the positive directions of East and North coordinate axes. Hence, care should be taken when determining x_P and y_P from *E,N* coordinates.

5.9 Fitting a Clothoid Spiral Transition Curve Between a Straight and a Circular Curve

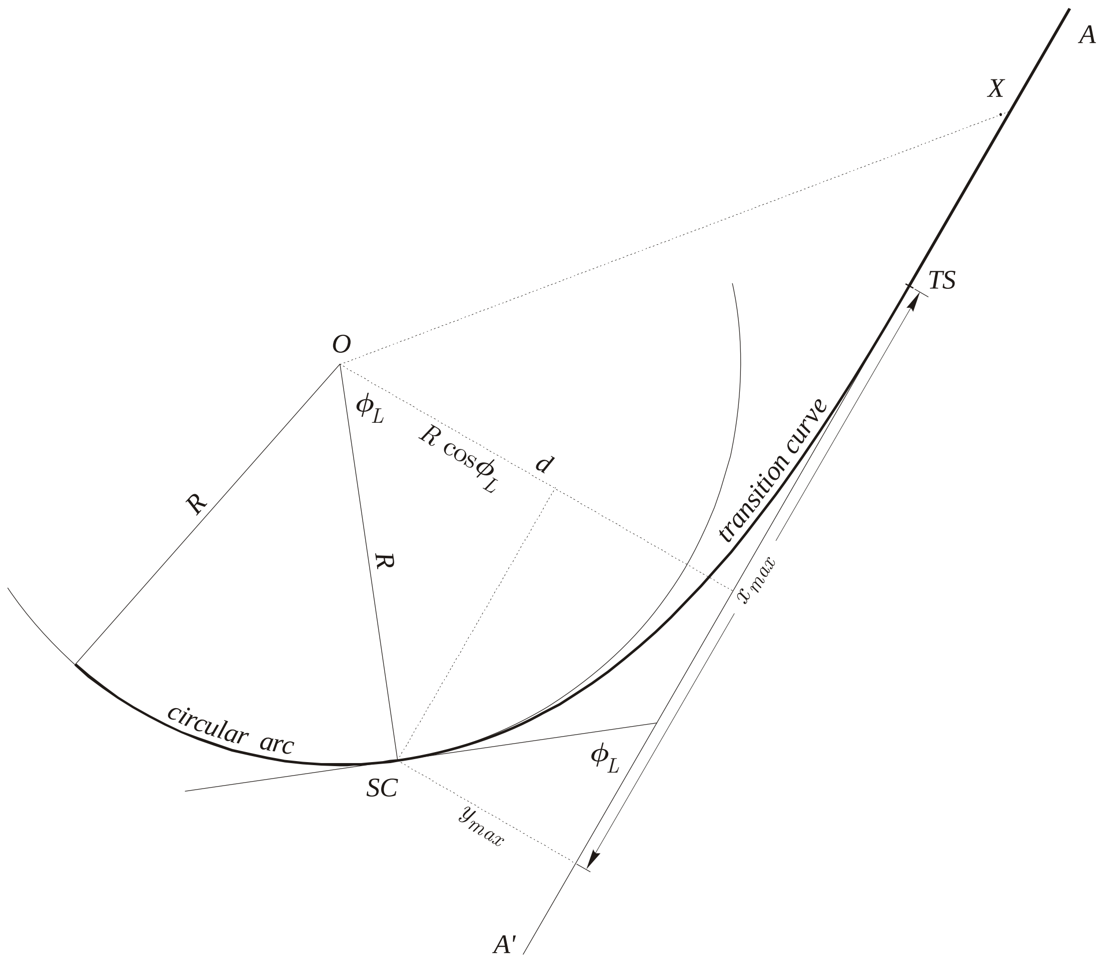


Figure 5.9

In Figure 5.9 a circular curve of radius *R* has a fixed centre at *O* having known coordinates. *AA'* is a straight of known bearing and it is desired to find the length *L* of a clothoid spiral transition curve that is tangential to the straight at *TS* and the circular curve at *SC*. The locations of *SC* and *TS* are unknown. *X* is a point on the straight *AA'* of known coordinates and the bearing and distance *OX* can be computed and then the perpendicular distance *d* from the centre *O* to the straight. The distance *d* must be greater than the radius *R*. From the diagram it can be seen that the spiral angle $\phi_L = L/(2R)$ (which is unknown) is also the angle at the centre *O* between the radial to *SC* and the perpendicular to the straight. The distance *d* is given by

$$d = R\cos\phi_L + y_{\max} = R\cos\left(\frac{L}{2R}\right) + y_{\max}$$

With the use of an *Excel* spreadsheet for computing clothoid spirals given parameters *L* and *R*, the length of the transition curve *L* can be determined by successively changing *L*, until the required distance *d* is obtained. $\phi_L = L/(2R)$ can be computed and the radial bearing to *SC* determined.

6. DESIGN CONSIDERATIONS FOR CIRCULAR CURVES AND TRANSITION CURVES

Circular curves and transition curves (clothoids) are uniquely defined if any two of their "properties" (or parameters) are fixed. For circular curves these two properties are usually selected from the following: radius, arc length, intersection angle, tangent length or chord length. For transition curves the properties are selected from minimum radius, length of curve, maximum tangential angle (also known as spiral angle) and shift.

Generally, in practical design of roads and railways, intersection angles of straight sections are predetermined by the overall layout and the problem is to design the connecting circular curves and transition curves to suit the expected traffic conditions. From a traffic viewpoint, the largest possible circular curves and the longest possible transition curves are most desirable, but restrictions usually arise due to the topography or site conditions and the cost. Therefore it is necessary to determine suitable *minima* for radii of circular curves and transition lengths for given traffic speeds. Since the speed of vehicles using a particular road or railway is a variable quantity (and is beyond the control of designers), "design speeds" are selected which satisfy some criteria. For instance, a design speed may be the speed where it is expected that it will not be exceeded by 85% of the vehicles using the road.

6.1 Minimum Radius for Circular Curves

Uniform Circular Motion

In Figure 6.1, a body of mass m is moving in a circular path of radius r at a constant velocity v. Such motion is known as uniform circular motion and the body has acceleration directed radially inwards towards the centre of the circle O. This acceleration is known as *centripetal acceleration* and in order for the body to have this acceleration it must be acted upon by a force $\mathbf{F}_c$ equal to its mass multiplied by its acceleration (Newton's second law $F = m \times a$). This force is known as the *centripetal force*. Equations for the centripetal acceleration and centripetal force can be derived in the following manner.

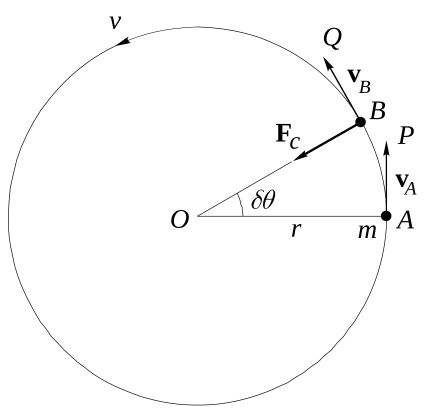
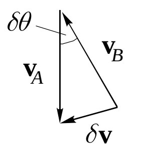


Figure 6.1

First, centripetal acceleration, remembering that

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{dv}{dt} \tag{i}$$

At A, the body of mass m has a velocity of magnitude v along the tangent AP. At B, its velocity is the same but its direction is now along BQ (the tangent at B).



The change in velocity is given by the vector subtraction $\overrightarrow{AP}$ from $\overrightarrow{BQ}$, i.e., $\delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$ and the angle between the vectors $\mathbf{v}_A$ and $\mathbf{v}_B$ is $\delta\theta$ (the angle between the radials OA and OB). In the limit, as B approaches A, the change in velocity can be considered as an arc of a circle of radius v subtending and angle $d\theta$. Thus the change in velocity is

$$dv = v\,d\theta \tag{ii}$$

Now, since velocity equals distance divided by time then $v = \frac{ds}{dt}$ where $ds = r\,d\theta$ and a re-arrangement gives the change in time as

$$dt = \frac{ds}{v} = \frac{r\,d\theta}{v} \tag{iii}$$

Substituting (ii) and (iii) into (i) and simplifying gives the centripetal acceleration

$$a = \frac{v^2}{r} \tag{6.1}$$

The centripetal force is found by Newton's second law ($F = m \times a$)

$$F_C = \frac{mv^2}{r} \tag{6.2}$$

An example of uniform circular motion and the resulting centripetal force is a stone on the end of a string rotating in a horizontal plane. The centripetal force in this instance is caused by the tension in the string.

For vehicles travelling at constant velocity around circular roadways or railway tracks, the centripetal force is caused by the constraining influence of the road pavement (friction) or the flanges of wheels on rail track.

Centrifugal force is a quantity peculiar to body moving in a circular path. It has the same magnitude as the centripetal force but points in the opposite direction. An occupant of a vehicle travelling around a circular curve "feels" the centrifugal force (acting in the opposite direction to the centripetal force) thrusting them against the side of the vehicle.

Superelevation and Friction

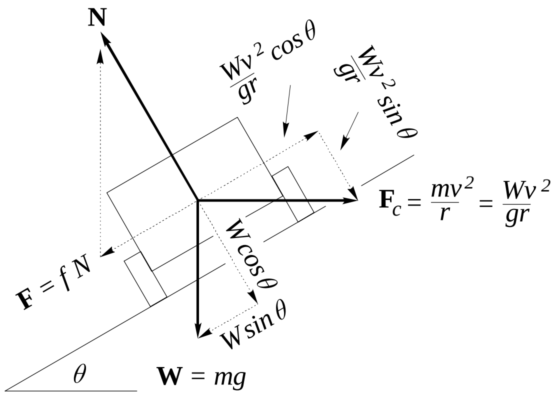


Figure 6.2

At any speed, in order to constrain a vehicle to follow a circular path, it is necessary to tilt or cant the road pavement or elevate the outer rail above the inner rail on rail track. This tilting or cant is known as superelevation and is used to reduce the effect of centrifugal force. In Figure 6.2, the superelevation is $e = \tan\theta$. In railway design, unsatisfactory superelevation will cause side thrust on the rails, spikes and sleepers and uneven wear on the rails. Wheels will ride up the outer rail and jump and carriages will tend to capsize. On roads, unsatisfactory superelevation will cause vehicles to slide and skid sideways.

In deciding how much superelevation to provide for a given velocity, too much may be as bad as too little. For **railways**, slow trains on steeply banked curves lurch inwards, whereas fast trains on curves with little or no superelevation would capsize or leave the track. One rule adopted is to provide superelevation for speed $V = \sqrt{\frac{1}{2}\left(V_{\max}^2 + V_{\min}^2\right)}$, which approximates the average speed of passenger trains, with an absolute maximum value of superelevation of 150 mm for a track gauge of 1.435 m.

For **roads**, the requirement is that maximum superelevation should not be so great as to disturb the stability of slow moving or stationary vehicles, particularly those carrying high loads. The maximum value adopted in Victoria for road design is 100 mm per metre or 1 in 10 (10%).

Referring to Figure 6.2, for road vehicles travelling at constant velocities around circular roadways, the centripetal force $\mathbf{F}_c$ is caused by the constraining influence of the road pavement (friction). The friction force $\mathbf{F} = f\ N$, which acts parallel to the road, is a function of **N**, the force normal to the road. *f* is the *coefficient of side frictional force* developed between the vehicle tyres and the road pavement. **W** is the weight (a force) and its magnitude *W* is equal to the vehicle mass *m* multiplied by the force of gravity *g*.

Within the limits of safe driving by an average driver, the coefficient of friction *f* ranges from 0.40 at 30 kph (kilometres per hour) to 0.11 at 110 kph. However, for reasons of passenger comfort, *f* should not exceed 0.20 and for design purposes, it is restricted to the range $0.11 \le f \le 0.19$.

The publication *Rural Road Design – Guide to the Geometric Design of Rural Roads* (AUSTROADS, Sydney, 1993) has a table of recommended maximum design values of *f* for sealed pavements, part of which is given below in Table 6.1

Design Speed *V* (kph)	Coefficient of Side Friction *f*
60	0.33
80	0.26
100	0.12
120	0.11
130	0.11

Table 6.1

In railway design, the coefficient of friction is ignored since the rails provide the entire constraining force.

Relationship between Superelevation (Cant) and Radius for given Velocity (Speed)

Speed is a measure of road (highway) design to which the geometrical properties of design are subordinated. The endeavour is to provide a continuous route that the road user can proceed along in comfort at uniform speed. Studies in Australia, have revealed that the majority of road users prefer to travel at speeds between 80 to 110 kph, and as a result of these studies have developed standards which are the basis for current highway design.

For roads, to make the thrust zero, road pavement must be superelevated until the components of the forces acting on the vehicle are balanced. Referring to Figure 6.2, resolving the forces acting on the vehicle into components parallel to the road gives

$$f N + W \sin\theta = \frac{Wv^2}{gr}\cos\theta \tag{6.3}$$

Resolving the forces acting on the vehicle into components normal to the road surface gives

$$N = \frac{Wv^2}{gr}\sin\theta + W\cos\theta \tag{6.4}$$

Substituting equation (6.4) into (6.3) and re-arranging gives

$$f\left(\frac{Wv^2}{gr}\sin\theta + W\cos\theta\right) = \frac{Wv^2}{gr}\cos\theta - W\sin\theta$$

Dividing both sides by $W\cos\theta$ and re-arranging

$$f\frac{v^2}{gr}\tan\theta + f + \tan\theta = \frac{v^2}{gr}$$

Now, the superelevation $e = \tan\theta$ hence

$$f\frac{v^2}{gr}e + f + e = \frac{v^2}{gr}$$

$$e + f = \frac{v^2}{gr}\left(1 - ef\right)$$

And the radius *r* is given by

$$r = \frac{v^2}{g}\left(\frac{1-ef}{e+f}\right) \tag{6.5}$$

For practical values of e and f the product ef is small (for $0.11 \le f \le 0.19$ and $e = 0.1$ then $0.011 \le ef \le 0.019$) and may be neglected giving

$$r = \frac{v^2}{g}\left(\frac{1}{e+f}\right) \tag{6.6}$$

In the equations above, v is m/s (metres per second). With V in kph (kilometres-per-hour) ($\text{m/s} \times 3.6 = \text{kph}$) and replacing r by R (the radius of the circular curve), and using $g = 9.8$ m/s as a representative value of the acceleration due to gravity, equation (6.6) becomes

$$R = \frac{V^2}{127\,(e+f)} \tag{6.7}$$

The publication *Rural Road Design – Guide to the Geometric Design of Rural Roads* (AUSTROADS, Sydney, 1993) has a table of Minimum Radii of Circular Curves based on Superelevation e and Side Friction f maxima. Part of this table is given below in Table 6.2

Vehicle Speed V (kph)	Superelevation e	Coefficient of Side Friction f	Minimum Radius R (m)
60	0.1	0.33	70
80	0.1	0.26	140
100	0.1	0.12	360
120	0.1	0.11	540
130	0.1	0.11	635

Table 6.2

The values in Table 6.2 have been computed using equation (6.7) and then rounded up to the nearest 5 metres. It is usual practice to adopt values greater than the minimum radius and to reduce superelevation and side friction below their maximum values.

6.2 Determination of Minimum Length of Transition Curve for Given Speed Values

Two methods may be adopted to determine lengths of transition curves L for given speeds V.

(i) Length is such that the full superelevation $e_{\max}$ is attained at a uniform time rate, say k metres per second (where k can vary from 0.03 to 0.06 m/s).

An equation for the length L can be developed in the following manner.

The time taken to travel the length L is $t = \frac{L}{v}$ where L is in metres, v is in metres per second and t is in seconds. $k = \frac{w\,e_{\max}}{t}$ metres per second where w is the road pavement width or railway track width and $e = \tan\theta$ is superelevation; $e_{\max}$ being the maximum value. Therefore $k = \frac{w\,e_{\max}\,v}{L}$ and by re-arrangement and using V in kph, noting that $v = \frac{V}{3.6}$

$$L = \frac{w\,e_{\max}\,V}{3.6k} \tag{6.8}$$

Equation (6.8) is used for computing lengths of transition curves for railway design where w is the width of the track, $w\,e_{\max}$ will be the height of the outer rail above the inner rail and values of k are adopted from empirical studies.

(ii) For riding comfort, the centripetal acceleration *a*, should increase gradually at a uniform rate, say *A* metres per second squared per second. Note: the units of *a* are m/s^2 and *A* are m/s^3.

An equation for the length *L* can be developed in the following manner.

As before, the time taken to travel the length *L* is $t = \frac{L}{v}$ where *L* is in metres, *v* is in metres per second and *t* is in seconds. The centripetal acceleration is $a = \frac{v^2}{R}$ (see equation (6.1) with *R* replacing *r*). If *A* is the uniform rate of increase in centripetal acceleration then $A = \frac{a}{t}$ and by substitution for *a* and *t* we obtain $A = \frac{v^3}{LR}$. By re-arrangement and using *V* in kph, noting that $v^3 = \frac{V^3}{(3.6)^3}$

$$L = \frac{0.0214V^3}{AR} \tag{6.9}$$

Equation (6.9) is used by *Vicroads* for computing lengths of road transition curves with the following values for *A*, the rate of change of radial acceleration

$V < 80$ kph $A = 0.60$
$80 \le V \le 120$ kph $A = 0.45$
$V > 120$ kph $A = 0.30$

Using these values for *A* with the minimum values for *R* in Table 6.2, some representative values for *L* are computed from equation (6.9) and given in Table 6.3 (rounded up to the nearest 5 m)

Vehicle Speed *V* (kph)	Minimum Radius *R* (m)	Rate of Change of Radial Acceleration *A* (m/s^3)	Length of Transition *L* (m)
60	70	0.60	110
80	140	0.45	175
100	360	0.45	135
120	540	0.45	155
130	635	0.30	250

Table 6.3

The values for *L* in Table 6.3 are far in excess of values adopted for the design of transition curves given in handbooks on the subject (see *Rural Road Design – Guide to the Geometric Design of Rural Roads,* AUSTROADS, 1993). In such cases, other considerations in the design come into play such as studies of driver behaviour. One should consider the fact that drivers often adopt cornering speeds based on what they can see of the road ahead. If the length of the transition "hides" the circular curve that drivers must negotiate then they may adopt an incorrect speed to safely negotiate the circular curve. To avoid this, transition curve lengths are often shorter than those derived from theoretical formula.

The paper by Leeming[2] has an interesting commentary on transition curves and superelevation. Leeming notes that the rate of change of radial acceleration is not the appropriate parameter to use in the design of transition curves. But, he makes a strong point that superelevation should not be introduced without a change in radius of curvature.

[2] Leeming, J. J., 1973, 'Road curvature and superelevation', *Survey Review*, Vol. XXII, No. 167, pp. 23-35.

6.3 Superelevation and Transition Curves

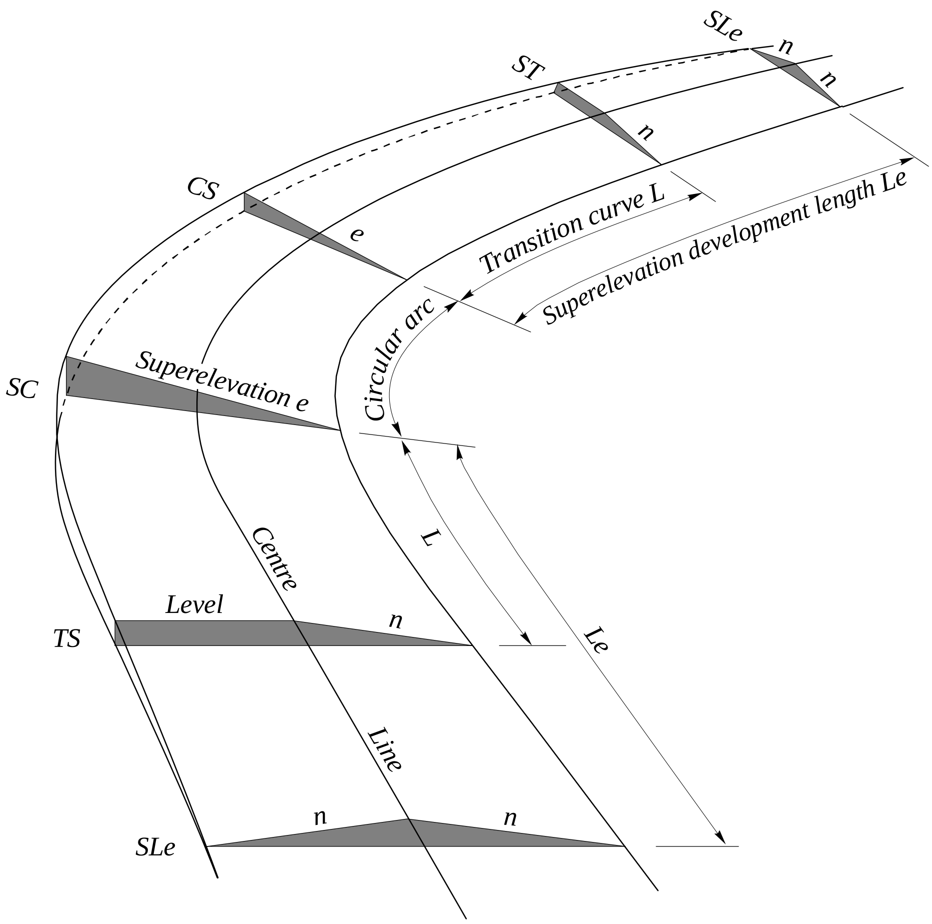


Figure 6.3

Figure 6.3 shows a schematic diagram of two straight sections of two-lane roadway joined by a circular curve with transition curves of length *L* joining the circular curve and the straights. Transition curves (clothoids) are also known as spirals and the tangent point of the straight and the spiral is known as *TS.* The common tangent to the spiral and the circular curve is *CS,* the common tangent to the circular curve and the spiral is *CS* and the spiral is tangential to the straight at *ST*. *n* is the cross-fall of the road (generally given in %) and *e* is the superelevation. At the start of the circular curve, *e* should be the maximum value adopted for the design. For a vehicle on the left-hand-side and travelling 'up' the road (from the bottom of the diagram) and turning to the right, the cross-fall *n* is negative (negative cant) and must change gradually to zero (level) at *TS.* At this point, superelevation begins (positive cross-fall or positive cant), which increases until it reaches its maximum value at the beginning of the circular curve. *Le* is the *length of superelevation development* and the point *SLe* is the point where the cross-fall starts to change as the vehicle approaches the transition curve. The distance between *TS* and *SLe* is usually dictated by the design velocity *V* and tables of values are given in design handbooks (eg, *Rural Road Design – Guide to the Geometric Design of Rural Roads*, AUSTROADS, 1993).